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Algebraic calculation of the Green function for a spinless charged particle in an external plane-wave electromagnetic field

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Abstract. The Green function for a spinless charged particle in the presence of an external plane-wave electromagnetic field is calculated by algebraic techniques in terms of the free-particle Green function.

1. Introduction

The Dirac equation for the electron in an external plane-wave electromagnetic field was solved by Volkov [1] and the Green function for this situation was obtained by Schwinger [2]. Although these problems were solved long ago they continue to attract attention from the viewpoint of physical applications in the treatment of the interaction of laser beams with electrons. Being exactly solvable, different techniques have been applied for rederiving the final results.

A similar situation exists in the case of the non-relativistic Coulomb problem which has been solved by several techniques, the most elegant being the algebraic one. The commutation relations satisfied by the conserved angular momentum $\boldsymbol{L}$ and the Runge-Lenz vector $\boldsymbol{A}$ lead to the calculation of the bound-state energy spectrum, wavefunctions, Green function and the scattering phase shifts.

In this paper we consider the application of an algebraic technique to calculate the Green function for a charged spinless particle in an external plane-wave electromagnetic field of the type considered by Schwinger. We show that it is possible to define canonically conjugate operators $\hat{x}_\mu$, $\hat{\pi}_\mu$ which include the effects of interactions and lead to a representation of the algebra of the restricted Poincaré group. We obtain the explicit form of the operators $\hat{x}_\mu$, $\hat{\pi}_\mu$ in terms of the operators x_μ, π_μ employed in the usual formulation of the problem. Our technique has several features in common with the earlier calculations of wavefunctions for the Klein-Gordon equation with interaction [3-5]. However, our results are valid for any gauge in contrast to the earlier ones.

This paper is organised as follows: in § 2 we formulate the problem in the coordinate gauge indicating how we can go to a general gauge, in § 3 we construct the conserved commuting momenta $\hat{\pi}_\mu$ which lead to the Green function, in § 4 the operators $\hat{x}_\mu$ conjugate to $\hat{\pi}_\mu$ are constructed and the Green function calculated in an alternative manner, and § 5 contains the conclusions.

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2. Formulation of the problem

Let $A_\mu(x)$ be the vector potential of the external electromagnetic field $F_{\mu\nu}(x)$ where†

$$F_{\mu\nu}(x) = \partial_\nu A_\mu - \partial_\mu A_\nu. \tag{1}$$

The field $F_{\mu\nu}(x)$ satisfies the free Maxwell equations

$$F^{\mu\nu},\ \nu = 0 \tag{2}$$

$$*F^{\mu\nu},\ \nu = 0 \tag{3}$$

where

$$*F^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{4}$$

Since the vector potential is arbitrary up to a gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Phi \tag{5}$$

we may choose a specific gauge that is convenient for a certain calculation and then go to a general gauge. If we choose the special gauge to be the coordinate gauge [7]

$$(x - x')^\mu A'_\mu(x) = 0 \tag{6}$$

where x' is a convenient reference point, then the gauge function Φ is given by

$$\Phi(x, x') = -P \int_{x'}^{x} A^\mu(y)\, \mathrm{d}y_\mu \tag{7}$$

where the symbol P indicates that the integration path is the straight line joining x' to x.

Let the external electromagnetic field have the form

$$F_{\mu\nu}(x) = f_{\mu\nu} F(\xi) \tag{8}$$

where $f_{\mu\nu}$ is a numerical antisymmetric tensor,

$$\xi = n \cdot x \qquad n^2 = 0 \tag{9}$$

and $F(\xi)$ is an arbitrary function of ξ. The field equations (2) and (3) imply that

$$n^\mu f_{\mu\nu} = 0 \tag{10}$$

$$n^\mu\ {*f_{\mu\nu}} = 0 \tag{11}$$

from which we get

$$*f_{\mu\lambda} f^\lambda_\nu = 0 \tag{12}$$

and with the choice of a normalisation factor

$$f_{\mu\lambda} f^\lambda_\nu = *f_{\mu\lambda}\ {*f^\lambda_\nu} = n_\mu n_\nu. \tag{13}$$

For the plane-wave field of equation (8) we have

$$A'_\mu(x) = f_{\mu\nu}(x - x')^\nu \chi(\xi, \xi') \tag{14}$$

where

$$2\chi + (\xi - \xi')\, \mathrm{d}\chi/\mathrm{d}\xi = F(\xi). \tag{15}$$

† We use the same notation as Bjorken and Drell [6].

Integrating and imposing the condition that χ is well behaved when $\xi \to \xi'$ we can write it in the form

$$\chi(\xi, \xi') = \frac{A(\xi)}{\xi - \xi'} - \frac{1}{(\xi - \xi')^2} \int_{\xi'}^{\xi} A(\eta)\, d\eta \tag{16}$$

where

$$F(\xi) = dA/d\xi. \tag{17}$$

We note that in this case the coordinate gauge is very convenient since, apart from equation (6), A'_μ also has the following two useful properties:

$$n^\mu A'_\mu = 0 \tag{18}$$

$$\partial^\mu A'_\mu = 0. \tag{19}$$

The Green function for a spinless charged particle in an external electromagnetic field satisfies the differential equation

$$(\pi^2 - m^2)\Delta(x, x') = \delta^{(4)}(x - x') \tag{20}$$

where

$$\pi_\mu = p_\mu - eA_\mu = i\partial_\mu - eA_\mu. \tag{21}$$

Denoting the Green function in the coordinate gauge by $\Delta'(x, x')$ we have

$$\Delta(x, x') = \Delta'(x, x') \exp(i\Phi(x, x')) \tag{22}$$

and $\Delta'(x, x')$ satisfies equation (20) with the understanding that $\pi_\mu = p_\mu - eA'_\mu$. This interpretation of π_μ is used in all subsequent calculations which use the coordinate gauge. We also note that the coincidence of the reference point x' in the coordinate gauge and the argument x' in the Green function is not essential. We have made this choice for simplicity.

A procedure for solving equation (20) is to put

$$\Delta'(x, x') = \int_0^\infty \exp(-im^2 s)\Delta'_s(x, x')\, ds \tag{23}$$

where the convergence factor is omitted for convenience. Thus we can determine $\Delta'(x, x')$ if we can find $\Delta_s(x, x')$ such that

$$i\Delta_s(x, x')|_{s \to 0+} = \delta^{(4)}(x - x') \tag{24}$$

and

$$(i\partial_s + \pi^2)\Delta'_s(x, x') = 0 \qquad s \neq 0. \tag{25}$$

Since the range of integration in equation (25) extends from 0 to ∞ we may impose the condition

$$\Delta'_s|_{s<0} = 0 \tag{26}$$

which allows us to substitute equations (24) and (25) by

$$(i\partial_s + \pi^2)\Delta'_s(x, x') = \delta(s)\delta^{(4)}(x - x'). \tag{27}$$

Thus $\Delta'_s(x, x')$ is formally the same as the Green function for a Schrödinger equation in a five-dimensional space with s as the 'time' and x_μ as the 'spatial' coordinates. We may identify the Hamiltonian by

$$H = -\pi^2. \tag{28}$$

When the field is not present one may follow the usual procedure of Fourier transformation and obtain the solution in the form

$$\Delta_{0s}(x, x') = \frac{\mathrm{i}\theta(s)}{(4\pi)^2 s^2} \exp\left(-\mathrm{i}\frac{(x-x')^2}{4s}\right) \tag{29}$$

where the suffix 0 indicates the absence of the external field.

3. Algebraic construction of the Green function

We remark that the differential equations (20) or (27) use the momenta π_μ which satisfy

$$[\pi_\mu, x_\nu] = \mathrm{i}g_{\mu\nu} \tag{30}$$

$$[\pi_\mu, \pi_\nu] = \mathrm{i}eF_{\mu\nu} \tag{31}$$

$$[\pi_\mu, H] = -2\mathrm{i}eF_{\mu\nu}\pi^\nu. \tag{32}$$

The last two equations indicate that π_μ are not very convenient to use. We show first that it is possible to construct new momenta $\hat{\pi}_\mu$ which reduce to p_μ when $e \to 0$, commute with H and commute among themselves. Due to the special type of electromagnetic field being considered we obtain

$$[n \cdot \pi, \xi] = 0 \tag{33}$$

$$[f_{\mu\nu}\pi^\nu, \xi] = 0 \tag{34}$$

$$[f_{\mu\nu}\pi^\nu, n \cdot \pi] = 0 \tag{35}$$

$$[n \cdot \pi, H] = 0 \tag{36}$$

$$[f_{\mu\nu}\pi^\nu, H] = -2\mathrm{i}en_\mu n \cdot \pi F(\xi) \tag{37}$$

$$[A(\xi), H] = 2\mathrm{i}n \cdot \pi F(\xi) \tag{38}$$

where there are no factor ordering problems. In fact all subsequent equations do not present factor ordering problems.

If we write

$$\hat{\pi}_\mu = \pi_\mu + \frac{ef_{\mu\nu}\pi^\nu A(\xi)}{n \cdot \pi} + \frac{e^2 A^2(\xi) n_\mu}{2n \cdot \pi} \tag{39}$$

we can verify by direct calculation that

$$[\hat{\pi}_\mu, H] = 0. \tag{40}$$

Equation (39) can be easily inverted to give

$$\pi_\mu = \hat{\pi}_\mu - \frac{eAf_{\mu\nu}\hat{\pi}^\nu}{n \cdot \pi} - \frac{e^2 A^2(\xi) n_\mu}{2n \cdot \pi} \tag{41}$$

where we used the relation

$$n \cdot \pi = n \cdot \hat{\pi} = n \cdot p. \tag{42}$$

If we write

$$\pi^\mu = \Lambda^\mu_\nu \hat{\pi}^\nu \tag{43}$$

then we have

$$\Lambda^{\mu}_{\nu} = g^{\mu}_{\nu} - \frac{eAf^{\mu}_{\nu}}{n \cdot \pi} + \frac{e^2 A^2 n^{\mu} n_{\nu}}{2(n \cdot \pi)^2}. \tag{44}$$

It is easy to verify that

$$\Lambda^{\mu}_{\rho} g_{\mu\nu} \Lambda^{\nu}_{\sigma} = g_{\rho\sigma} \tag{45}$$

so that Λ is a Lorentz transformation. One can also verify that

$$\Lambda^{\mu}_{\nu} n^{\nu} = n^{\mu} \tag{46}$$

$$\Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} f^{\rho\sigma} = f^{\mu\nu}. \tag{47}$$

Further, the following commutation rules hold:

$$[\pi^{\rho}, \Lambda^{\nu}_{\sigma}] = -\frac{\mathrm{i}eFf^{\nu}_{\sigma}}{n \cdot \pi} + \frac{\mathrm{i}e(\mathrm{d}/\mathrm{d}\xi)(A^2) n^{\rho} n^{\nu} n_{\sigma}}{2(n \cdot \pi)^2} \tag{48}$$

$$[\Lambda^{\mu}_{\nu}, \Lambda^{\rho}_{\sigma}] = 0. \tag{49}$$

Using the last four equations and equation (11), we obtain

$$[\hat{\pi}^{\mu}, \hat{\pi}^{\nu}] = 0. \tag{50}$$

Since Λ is not a constant-parameter Lorentz transformation there is no contradiction in having equations (31), (43) and (50) holding simultaneously.

Also using equations (43) and (48) one can see that

$$H = -\hat{\pi}^2. \tag{51}$$

Next recalling that $\hat{\pi}_{\mu} \to p_{\mu}$ when $e \to 0$ and

$$[p_{\mu}, p_{\nu}] = 0 \tag{52}$$

we guess that

$$\hat{\pi}_{\mu} = U p_{\mu} U^{-1} \tag{53}$$

where U is an unitary transformation depending on the interaction. To construct U it is convenient to rewrite $\hat{\pi}_{\mu}$ in the form

$$\hat{\pi}_{\mu} = p_{\mu} + \frac{e}{n \cdot p}[A - (\xi - \xi')\chi] f_{\mu\nu} p^{\nu} - \frac{eA' \cdot p}{n \cdot p} n_{\mu} + \frac{e^2}{2n \cdot p}[A^2 - 2A(\xi - \xi')\chi] n_{\mu}. \tag{54}$$

The transformation

$$U_1 = \exp\left(-\frac{\mathrm{i}e^2}{2n \cdot p}\Omega(\xi, \xi')\right) \tag{55}$$

gives

$$U_1 p_{\mu} U_1^{-1} = p_{\mu} - \frac{e^2}{2n \cdot p} \frac{\mathrm{d}\Omega}{\mathrm{d}\xi} n_{\mu}. \tag{56}$$

Also if

$$U_2 = \exp\left(-\frac{\mathrm{i}e}{n \cdot p}\Gamma(\xi, \xi') A' \cdot p\right) \tag{57}$$

then we have

$$U_2 p_\mu U_2^{-1} = p_\mu + \frac{e\Gamma\chi}{n \cdot p} f_{\mu\nu} p^\nu - \frac{eA' \cdot p}{\chi n \cdot p} \frac{\mathrm{d}}{\mathrm{d}\xi}(\Gamma\chi) n_\mu + \frac{e^2\chi^2\Gamma^2}{2n \cdot p} n_\mu. \tag{58}$$

Hence

$$U_1 U_2 p_\mu (U_1 U_2)^{-1} = p_\mu + \frac{e\Gamma\chi}{n \cdot p} f_{\mu\nu} p^\nu - \frac{eA' \cdot p}{\chi n \cdot p} \frac{\mathrm{d}}{\mathrm{d}\xi}(\Gamma\chi) n_\mu + \frac{e^2}{2n \cdot p}\left(\Gamma^2\chi^2 - \frac{\mathrm{d}\Omega}{\mathrm{d}\xi}\right) n_\mu. \tag{59}$$

Comparing equations (54) and (59) we get $U = U_1 U_2$ if

$$\Gamma\chi = A - (\xi - \xi')\chi \tag{60}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\xi} = \Gamma^2\chi^2 - A^2 + 2A(\xi - \xi')\chi. \tag{61}$$

Integrating equation (61) with the help of equation (60) we obtain

$$\Omega(\xi, \xi') = \int_{\xi'}^{\xi} A^2(\eta)\, \mathrm{d}\eta - \frac{1}{\xi - \xi'}\left(\int_{\xi'}^{\xi} A(\eta)\, \mathrm{d}\eta\right)^2. \tag{62}$$

Next, using the fact that $\Delta'(x, x')$ is a matrix element of the resolvent operator $(\pi^2 - m^2)^{-1}$ we obtain

$$\Delta'(x, x') = U_x \Delta_0(x, x') U_{x'}^{-1} \tag{63}$$

where U_x is U in the x representation as determined above and $U_{x'}$ means U in the limit $x \to x'$. This gives

$$\Delta'_s(x, x') = U_x \Delta_{0s}(x, x') U_{x'}^{-1}. \tag{64}$$

Next, due to our choice of the reference point in the coordinate-gauge vector potential we have

$$U_{x'} = I \tag{65}$$

and

$$A' \cdot p\Delta_{0s}(x, x') = 0 \tag{66}$$

so that U_2 can be replaced by the identify transformation in equations (63) and (64).

In fact one can verify directly that

$$UA' \cdot pU^{-1} = A' \cdot p \tag{67}$$

so that

$$A' \cdot p\Delta'_s(x, x') = 0. \tag{68}$$

Further, equation (29) gives

$$2n \cdot p\Delta_{0s} = \frac{(\xi - \xi')}{s} \Delta_{0s}. \tag{69}$$

Putting all this information together we obtain equation (64) in the form

$$\Delta'_s(x, x') = \Delta_{0s}(x, x') \exp\left(-\frac{\mathrm{i}e^2 s}{\xi - \xi'} \Omega(\xi, \xi')\right) \tag{70}$$

which leads to Schwinger's result [2] for the Green function $\Delta(x, x')$.

4. An alternative construction of the Green function

In this section we exploit the observation that, in terms of the momenta $\hat{\pi}_\mu$, H is simply a free-particle Hamiltonian. Since the Green function Δ' is a matrix element of the resolvent operator $(\pi^2 - m^2)^{-1}$ one can write down immediately $\Delta'(\hat{x}, \hat{x}')$ where the arguments $\hat{x}, \hat{x}'$ refer to the eigenvalues of a position operator $\hat{x}_\mu$ conjugate to $\hat{\pi}_\mu$ were it to exist. To find $\Delta'(x, x')$ it is only necessary to construct the transformation function from the $\hat{x}$ to x representation. An immediate candidate for $\hat{x}_\mu$ is

$$\hat{x}_\mu = U x_\mu U^{-1} \tag{71}$$

which can be explicitly written in the form

$$\hat{x}_\mu = x_\mu + \frac{e\Gamma\chi(\xi-\xi')}{(n\cdot p)^2} f_{\mu\nu}\pi^\nu + \frac{e^2}{2(n\cdot p)^2}[\Gamma^2\chi^2(\xi-\xi') + 2\Gamma\chi^2(\xi-\xi') - \Omega]n_\mu. \tag{72}$$

On the other hand, one may directly calculate

$$[\hat{\pi}_\mu, x_\nu] = \mathrm{i}g_{\mu\nu} - \frac{ieA}{(n\cdot\pi)^2} n_\mu f_{\nu\lambda}\pi^\lambda - \frac{ie^2A^2}{2(n\cdot\pi)^2} n_\mu n_\nu \tag{73}$$

which together with the condition

$$[\hat{\pi}_\mu, \hat{x}_\nu] = \mathrm{i}g_{\mu\nu} \tag{74}$$

suggests the form for $\hat{x}_\mu$ as

$$\hat{x}_\mu = x_\mu + \frac{eR(\xi,\xi')}{(n\cdot\pi)^2} f_{\mu\nu}\pi^\nu + \frac{e^2S(\xi,\xi')}{2(n\cdot\pi)^2} n_\mu \tag{75}$$

where we must impose the conditions

$$\mathrm{d}R/\mathrm{d}\xi = A(\xi) \tag{76}$$

$$\mathrm{d}S/\mathrm{d}\xi - 2R\,\mathrm{d}A/\mathrm{d}\xi = A^2. \tag{77}$$

Equations (72) and (75) are the same if

$$R(\xi) = \int_{\xi'}^{\xi} A(\eta)\,\mathrm{d}\eta \tag{78}$$

$$S(\xi) = -\int_{\xi'}^{\xi} A^2(\eta)\,\mathrm{d}\eta + 2A(\xi)\int_{\xi'}^{\xi} A(\eta)\,\mathrm{d}\eta \tag{79}$$

which are consistent with equations (76) and (77). The fact that the direct calculation of $\hat{x}_\mu$ gives the same result as that in equation (71) indicates that the form of $\hat{x}_\mu$ is essentially unique. Clearly we have

$$[\hat{x}_\mu, \hat{x}_\nu] = 0. \tag{80}$$

Also if

$$\hat{x}_\mu|\hat{x}_1\rangle = x_{1\mu}|\hat{x}_1\rangle \tag{81}$$

we get

$$|\hat{x}_1\rangle = U|x_1\rangle \tag{82}$$

where

$$x_\mu|x_1\rangle = x_{1\mu}|x_1\rangle. \tag{83}$$

Due to our choice of reference point in the form of the vector potential

$$|\hat{x}'\rangle = |x'\rangle. \tag{84}$$

Thus

$$\begin{aligned}\Delta'(x, x') &= \int \langle x|\hat{y}\rangle\left\langle \hat{y}\left|\frac{1}{\hat{\pi}^2 - m^2}\right|\hat{x}'\right\rangle \mathrm{d}^4y \\ &= \int \langle x|U|y\rangle\Delta_0(y, x')\,\mathrm{d}^4y \\ &= U_x\Delta_0(x, x').\end{aligned} \tag{85}$$

Due to the fact that the same unitary transformation U relates π_μ, $\hat{\pi}_\mu$ and x_μ, $\hat{x}_\mu$ we do not get a new form for $\Delta'(x, x')$. On the other hand, it is satisfactory that the problem with external interaction has been transformed into a free-particle problem by the unitary transformation.

We also note that if we define

$$\hat{M}_{\mu\nu} = \hat{x}_\mu\hat{\pi}_\nu - \hat{x}_\nu\hat{\pi}_\mu \tag{86}$$

the operators $\hat{\pi}_\mu$, $\hat{M}_{\mu\nu}$ generate a representation of the algebra of the restricted Poincaré group. The Pauli-Lubanski operator

$$W^\lambda = \tfrac{1}{2}\varepsilon^{\lambda\mu\nu\sigma}\hat{M}_{\mu\nu}\hat{\pi}_\sigma \tag{87}$$

is identically zero so that $W^2 = 0$; hence the spin is zero. The propagator has a pole at $\hat{\pi}^2 = m^2$ so that we have a mass-m, spin-0 representation of the Poincaré algebra. This, of course, does not mean that the observed mass is m since $\hat{\pi}$ includes the interaction. The final form of the propagator in equation (70) indicates that the observed mass is $m + \Delta m$ where $\Delta m > 0$ since $\Omega > 0$.

5. Conclusion

We have constructed the Green function for a spin-zero particle in an external plane-wave electromagnetic field of a general type by relating it to the free-particle Green function through a unitary transformation U. Starting from the variables x_μ, π_μ we have constructed new canonical variables $\hat{x}_\mu$, $\hat{\pi}_\mu$ with $\hat{\pi}_\mu$ behaving like free-particle momenta. The unitary transformation relates the pair $\hat{x}_\mu$, $\hat{\pi}_\mu$ to x_μ and p_μ. The variables $\hat{x}_\mu$, $\hat{\pi}_\mu$ include the effects of the interaction and can be used to construct a fixed-mass spin-zero representation of the Poincaré algebra. Alternatively, the Green function viewed as a matrix element of the resolvent operator is simply the free-particle propagator in a representation in which $\hat{x}_\mu$ is diagonal. The multiplicative factor appearing in the Green function is due to the fact that a matrix element in the x representation is to be calculated and x_μ and $\hat{x}_\mu$ are not mutually compatible.

It may also be noted that our results can be generalised to the case of arbitrary polarisation. Instead of equation (8) we could start with the external field

$$F_{\mu\nu}(x) = \sum_{a=1}^{2} f^a_{\mu\nu}F^a(\xi) \tag{88}$$

where one can always choose the normalisation

$$f^a_{\mu\lambda}f^{\lambda b}_{\nu} = \delta^{ab}n_\mu n_\nu \tag{89}$$

corresponding to the situation of perpendicular polarisations. All the subsequent equations are modified in an obvious manner. The Lorentz transformation Λ of equation (43) is replaced by a product of two commuting Lorentz transformations, one for each polarisation. Similarly, U will be replaced by a product of two commuting unitary transformations. Equation (70) will now have $\Omega_1+\Omega_2$ in place of Ω.

We also mention that a direct solution of the differential equation for the Green function is possible at the cost of the additional assumption [8] that

$$\Delta'_s(x, x') = \Delta_{0s}(x, x')\Sigma_s(\xi, \xi') \tag{90}$$

which amounts to imposing equation (68). The algebraic approach provides a justification for this condition.

We did not explicitly consider the solutions of the Klein-Gordon equation. Our technique yields a complete set of solutions of the Volkov type in the coordinate gauge with $\hat{\pi}_\mu$ diagonal. The limitation to the coordinate gauge can be easily removed by taking into account the phase factor of equation (22). This allows us to write solutions in the form:

$$\psi(x, x') = [\exp(\mathrm{i}\Phi(x, x'))]U_x\psi_0(x) \tag{91}$$

where ψ_0 satisfies the free-particle Klein-Gordon equation. To compare with the results of earlier calculations [3-5] we use the gauge

$$A_\mu(x) = \varepsilon_\mu A(\xi) \tag{92}$$

which gives

$$\begin{aligned}[\exp(\mathrm{i}\Phi(x, x'))]U_x &= \exp\left\{-\frac{\mathrm{i}e}{n\cdot p}\int_{\xi'}^{\xi} A\cdot p\,\mathrm{d}\eta \right.\\ &\left. -\frac{\mathrm{i}e^2}{2n\cdot p}\left[\int_{\xi'}^{\xi} A^2\,\mathrm{d}\eta - \frac{1}{\xi-\xi'}\left(\int_{\xi'}^{\xi} A\,\mathrm{d}\eta\right)^2\right]\right\}.\end{aligned} \tag{93}$$

The limit $\xi'\to-\infty$ with $\int_{\xi'}^{\xi} A\,\mathrm{d}\eta$ finite reproduces the results of these calculations.

Finally we note that, although our final results are implicitly contained in Schwinger's paper [2], we have obtained them without having to solve the complicated equations of motion for the operators x_μ and π_μ in the Heisenberg picture.

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